Unexpected temporal tunnel-effects predicted by the inner Schwarzschild solution of Einstein's field equation of General Relativity

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Abstract: The inner Schwarzschild solution of Einstein's field equation of General Relativity allows the determination of a world line (on a **t,r**-chart) of a photon or of Alice (traveling at a local speed of almost the local speed of light, say 0.999998 **c**) that are falling towards the center of a solid spherical body through a vertical shaft. In the special case in which the local escape velocity is as high as **c** only at the center (and is lower elsewhere), all the gravitating mass of the Black Hole sits outside of the Black Hole proper. This distinguishes these Black Holes from usual ones (described by the *outer* Schwarzschild solution), in which the gravitating mass is concentrated at the center (at **r**=0) deep inside the Black Hole proper. Kruskal-Szekeres charts (taylored for the *outer* Schwarzschild solution) are not applicable to the inner Schwarzschild solution. Therefore any obstruction to a complete traverse of the spherical body by freely falling/rising Alice does not exist. Regarding solid spherical bodies for which 1>= $\mathbf{r_s/R_0}$ >8/9, it turns out that Alice's world line has a section in which her proper time **tau** is reversed relative to coordinate time **t** (which is the time of the distant observer Bob). That is to say: **dtau/dt** is a negative, small but non-vanishing number. When it comes to solid spherical bodies for which $\mathbf{r_s/R_0}$ is <= 8/9 (like planet Earth), Alice is tunneling an interval of coordinate time into the past at the center of the solid spherical body. That is to say: **dtau/dt**=0.

Key words: Inner Schwarzschild solution, tunnel-effect

1) The world line of a photon falling through a radial shaft – drilled across a massive, spherical body that constitutes a Black Hole – according to the inner Schwarzschild solution

a) The inner Schwarzschild solution of Einstein's field equation reads [in polar coordinates, see Motschmann (2020); see also Fließbach, T. (2003), p. 218, and also Xiaochun, M. (2011), pp. 109-116, where $\mathbf{R_0^3/r_s}$ is replaced by $\mathbf{R^2}$]:
(1)

$$c^2 d\tau^2 = \frac{1}{4}\left(3\sqrt{1-\frac{r_s}{R_0}} - \sqrt{1-\frac{r^2 r_s}{R_0^3}}\right)^2 c^2 dt^2 - \frac{1}{1-\frac{r^2 r_s}{R_0^3}} dr^2 - r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

It applies to the interior of a sphere which is homogeneously filled with matter. $\mathbf{R_0}$ is the radius of that spherical body (defined as circumference of a circle divided by 2**pi**). The radius $\mathbf{r_s}$ (Schwarzschild radius) is not the radial distance at which the local escape velocity is **c**, but is simply equal to **2GM/c²** (with **G** being Newton's gravitational constant, **M** being the mass of the spherical body, **c** being the local speed of light). The angle **theta** denotes "latidude", the angle **phi** denotes "longitude". The parameter **tau** is the proper time of a stationary or moving observer Alice in the gravity field (with **d tau** being the proper time interval between two point events that occur at the same spatial location for that observer), **t** is the time of a distant, stationary observer Bob who sits "outside" of the gravity field, **r** is the radial distance of a point event, measured as circumference (that touches said point) of a circle around the center of the sphere, divided by 2 pi.

For an observer Alice who is traveling at a speed very close to that of light **c** (in Bob's frame of reference), say, at 0.999998 **c**, or for a photon traveling at the speed of **c**, the temporal interval of Alice's (or the photon's) proper time **Delta tau** during which moving objects have passed by Alice (or by the photon) who considers herself at rest in her own frame of reference is very close to zero (or zero, respectively). So is the quotient **d tau/dt**. Consequently, the left-hand side of (1) can be set to zero. Presuming that, in Bob's **t**,**r**, **theta**, **phi**-frame of reference, spatial motions of Alice (or of a photon) occur in the radial or anti-radial direction only (so that **d theta** and **d phi** are both zero), (1) converts into:
(2)

$$v_{light} = \frac{dr}{dt} = \pm\frac{1}{2}\left(3\sqrt{1-\frac{r_s}{R_0}} - \sqrt{1-\frac{r^2 r_s}{R_0^3}}\right)\left(1-\frac{r^2 r_s}{R_0^3}\right)^{1/2} c$$

The polar coordinate system used for presenting the Schwarzschild solution (1) can now be replaced by a two-dimensional Cartesian coordinate system with **+-r** as the abscissa and **+-t** as the ordinate.

One realizes: If $\mathbf{r_s}=\mathbf{R_0}$ (in this case $\mathbf{r_s}$ is identical with the radial distance **r** at which the local escape velocity is **c**), the speed of light at **r=0** is +-0,5 **c**.

Morever, from (2) follows for an ingoing photon that starts its journey at $\mathbf{+r}=+\mathbf{R_0}$ and ends up at **r**=0 (with $\mathbf{t_{r\_1}}$ being the point in Bob's time when the ingoing photon is at $\mathbf{+r}=+\mathbf{R_0}$, and with $\mathbf{t_{r\_0}}$ being the point in Bob's time when the ingoing photon has reached **r**=0):
(3)

$$t_{r_0} = t_{r_1} + \Delta t$$

$$\Delta t = \int_{r=R_0}^{r=0} d[t(r)] = \int_{r=R_0}^{r=0} \frac{-2}{\left(3\sqrt{1-\frac{r_s}{R_0}} - \sqrt{1-\frac{r^2 r_s}{R_0^3}}\right)\left(1-\frac{r^2 r_s}{R_0^3}\right)^{1/2} c}\, dr$$

If $\mathbf{R_0}$=+1, **c**=+1, and $\mathbf{r_s}/\mathbf{R_0}$ = +0,9 = +8.1 / 9, and if an auxiliary summand **epsilon** is introduced that approaches zero, (3) turns into (see Fig. 1):
(4)

$$\Delta t = \int_{r=1}^{r=0} d[t(r)] = \int_{r=1}^{r=0} \frac{-2}{\left(3\sqrt{0.1} - \sqrt{1-0.9r^2}\right)\left(1-0.9r^2\right)^{1/2}}\, dr$$

$$= -\frac{20}{3}\Big[\ln\Big(\frac{|\sqrt{10-9r^2}+3(\sqrt{10}-3)r-\sqrt{10}|}{|r+\varepsilon|}\Big) - \ln\Big(\frac{|\sqrt{10-9r^2}-3(\sqrt{10}-3)r-\sqrt{10}|}{|r+\varepsilon|}\Big)\Big]_{r=1}^{r=0} = -3.05430$$

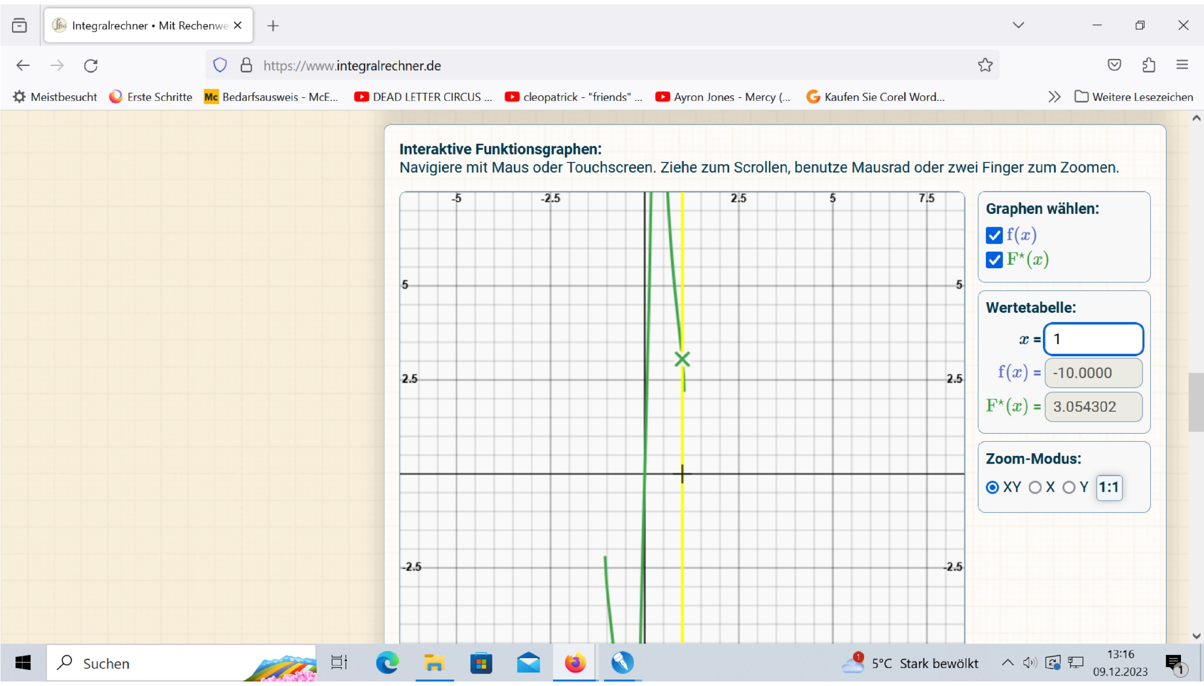

Fig. 1: A **t,r**-diagram showing the world line of Alice (traveling almost at the speed of light) who is traversing a spherical body from **r**(=**x**)=+1 to **r**(=**x**)= -1 with $\mathbf{r_s/R_0}$= 0.9 (generated online at www.integralrechner.de). The local escape velocity is +-**c** at **r**=+-0.3 (with $\mathbf{R_0}$=+-1).

At (roughly) **r**= +-0.3, the integrand goes to infinity. That's where the local escape velocity is **+-c**.

When beginning its trip at **t**=+3.05430, **r**=+1, the photon (or fast traveling Alice) "first" requires an infinitely long temporal interval in Bob's time **t** (Bob sits outside the gravity field) for closely approaching **r**=+0.3 in Bob's very distant future. It "then" travels backward into Bob's past in order to reach **r**=0 at Bob's time **t**=0. In other words: Alice's time is inverted in Bob's frame of reference between **r**=+0.3 and **r**=0 as a net result.

The certainty of a time-inversion is provided by the following reflection: Undoubtedly (as a result of the relativity principle, see below), Alice makes it all the way from **r**=+$\mathbf{R_0}$=+1 to **r**=0 in a short interval of her proper time **tau**. Hence, when setting marks on Alice's world line (in the **t,r**-diagram) that stand for individual tickings of Alice's clock, those (unequidistant) marks must be consecutively numbered along the world line all the way from **r**=+$\mathbf{R_0}$=+1 to **r**=0. The consecutive numbering even overcomes the "barrier" at **r**=+0.3 that cannot be crossed by Alice in Bob's frame of reference (but doesn't present an obstacle in Alice's frame of reference). This means: Between **r**=+0.3 and **r**=0, the arrow of time along Alice's world line is inverted relative to Bob's. In other words: **d tau/dt** is a negative, small but non-vanishing number.

b) To summarize: With regard to spherical bodies for which 1>= $\mathbf{r_s/R_0}$ >8/9, the inner Schwarzschild solution yields a world line of fast (almost at the local speed of **c**) falling Alice with a singularity, and also a time-reversed section between the singularity and the center of the sphere. The integral of **dt** (that determines the shape of the world line from $\mathbf{+R_0}$ all the way to the center and further on to $\mathbf{-R_0}$) is not divergent, if the integral is formed between the limit **r**=+1 and the limit **r**=0.

2) The traverse of a spherical body – that constitutes a Black Hole – by a photon according to the inner Schwarzschild solution

a) A special situation presents itself if $\mathbf{r_s}$ = 8/9 $\mathbf{R_0}$. Then the radial distance **r** (from the center) at which the local escape velocity is **c** is just zero (and less than zero anywhere else). Then (3) turns into:
(5)

$$\Delta t = \int_{r=+1}^{r=-1} d[t(r)] = \int_{r=+1}^{r=-1} \frac{-2}{\left(1 - \sqrt{1-\frac{8}{9}r^2}\right)\left(1-\frac{8}{9}r^2\right)^{1/2}} dr = \left[\frac{-6r}{\sqrt{9-8r^2}-3}\right]_{r=+1}^{r=-1} = -6.0000$$

The time-inverted section of Alice's world line, which was infinitely long, but did not quite coincide with the vertical axis of the **t,r**-diagram in the previous case (of $\mathbf{r_s/R_0}$=0.9), can now be considered to do so.

In order to be at **r**=0 at time **t**=0, fast traveling Alice (or the photon) now has to pass by **r**=+1 at exactly **t**=+3 (see Fig. 2). As before, fast traveling Alice "first" requires an infinitely long temporal interval in Bob's time **t** for closely approaching **r**=0 in Bob's very distant future. Alice "then" almost stands still (in Bob's frame of reference) for an infinitely long period of Bob's time. But during that infinitely long period of Bob's time **t**, the arrow of time along Alice's vertical world line is again inverted in Bob's frame of reference. (One has to recall that both the inner and the outer Schwarzschild solution *presuppose* that the central gravitating spherical mass has been and will be there for ever.)

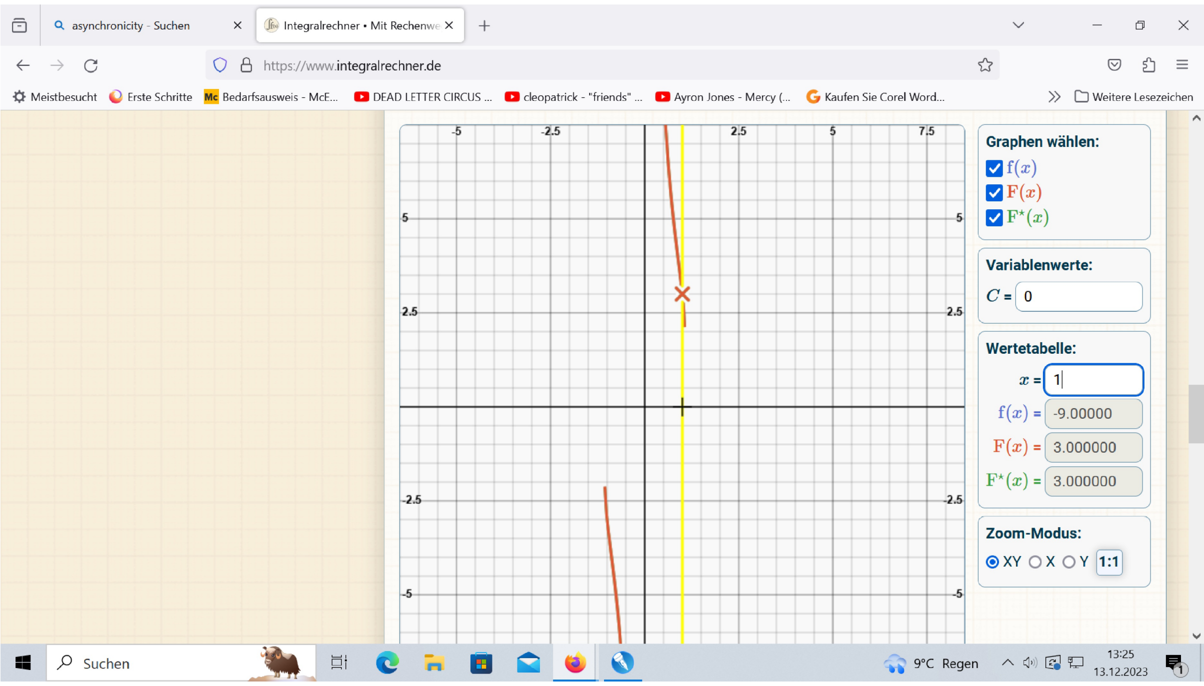

Fig. 2: A **t,r**-diagram showing the world line of Alice (traveling almost at the speed of light) who is traversing a spherical body from **r**(=**x**)=+1 to **r**(=**x**)= -1 with $\mathbf{r_s/R_0}$=8/9 (generated online at www.integralrechner.de).The local escape velocity is **+-c** at **r**=0 (with $\mathbf{R_0}$=+-1).

b) Will Alice be able to leave the spherical body on the other side, that is to say, can she reach **r**=**-**$\mathbf{R_0}$=-1? Yes, she can. A Kruskal-Szekeres chart, which is commonly thought to prove that such a thing is impossible in a Black Hole whose mass is concentrated at a mathematical point at **r**=0, is not applicable here. In other words: The Kruskal-Szekeres chart is not applicable when it comes to the *inner* Schwarzschild solution, neither if $\mathbf{r_s}$ /$\mathbf{R_0}$ = 8.1 / 9 =0.9, nor if $\mathbf{r_s}$ /$\mathbf{R_0}$=8/9, and not even if $\mathbf{r_s}$ /$\mathbf{R_0}$ = 1. It can therefore be left undecided whether the common way of utilizing Kruskal-charts (in the context of the *outer* Schwarzschild solution) is the correct one, or whether *different* quadrants of the chart – rather than one and the same quadrant – must be used for ingoing and outgoing objects [see Trupp (2020) and Trupp (2024)].

In the case of $\mathbf{r_s}$ /$\mathbf{R_0}$=8/9, all the gravitating mass of what still constitutes a Black Hole sits outside of the Black Hole proper, that is, at places where the local escape velocity is below **c**. Alice (presumed to be traveling at a speed of 0.999998 **c**) is thus capable of traversing the spherical body through a shaft that has been drilled all across the spherical body, even if this were impossible (as is commonly assumed) in cases of ordinary Black Holes (in which all the gravitating mass sits in their interiors).

More precisely: Having passed by **+r**=**+x**= **+**$\mathbf{R_0}$, it will take 6 temporal units of Bob's time **t** for rushing Alice until she will have reached the position **-r**=**-x**=**-**$\mathbf{R_0}$ on the other side of the spherical body. But these 6 (net) units of Bob's time are, as a net result, *inverted* units of time, as Alice takes a leap into Bob's past when at **r**=0. In mathematical terms, we find for that section of Alice's world line that coincides with the vertical axis of the **t,r**-diagram: **dtau/dt**=0. But this is identical to saying that Alice "tunnels" an infinitely long period of

Bob's time at **r**=0.

3) The traverse of a spherical body that is not a Black Hole, and the dilemma which arises for the relativity principle

a) What if $\mathbf{r_s/R_0} < 8/9$? Even if $\mathbf{r_s/R_0}$ is only a tiny bit smaller than 8/9, say 7.999999999998 / 9, the local escape velocity at **r**=0 falls a tiny bit short of **c**, and Alice's (or the photon's) world line appears to have a completely different shape. In case of $\mathbf{r_s/R_0}$= 7/9 (for instance), Equation (3) turns into (the auxiliary summand **epsilon** approaches zero):
(6)

$$\Delta t = \int_{r=1}^{r=0} d[t(r)] = \int_{r=1}^{r=0} \frac{-2}{\left(3\sqrt{1-\frac{7}{9}} - \sqrt{1-\frac{7}{9}r^2}\right)\left(1-\frac{7}{9}r^2\right)^{1/2}}\, dr$$

$$= \left[\frac{12\arctan\dfrac{\sqrt{\sqrt{2}+1}\,\left(\sqrt{9-7r^2}-3\right)}{\sqrt{\sqrt{2}-1}\,\sqrt{7}r+\varepsilon}}{\sqrt{\sqrt{2}-1}\,\sqrt{\sqrt{2}+1}\,\sqrt{7}}\right]_{r=1}^{r=0} = 4.38173$$

Apparently, there is no longer any time-inverted section of Alice's world line (see Fig. 3 for $\mathbf{r_s/R_0}$= 7/9).

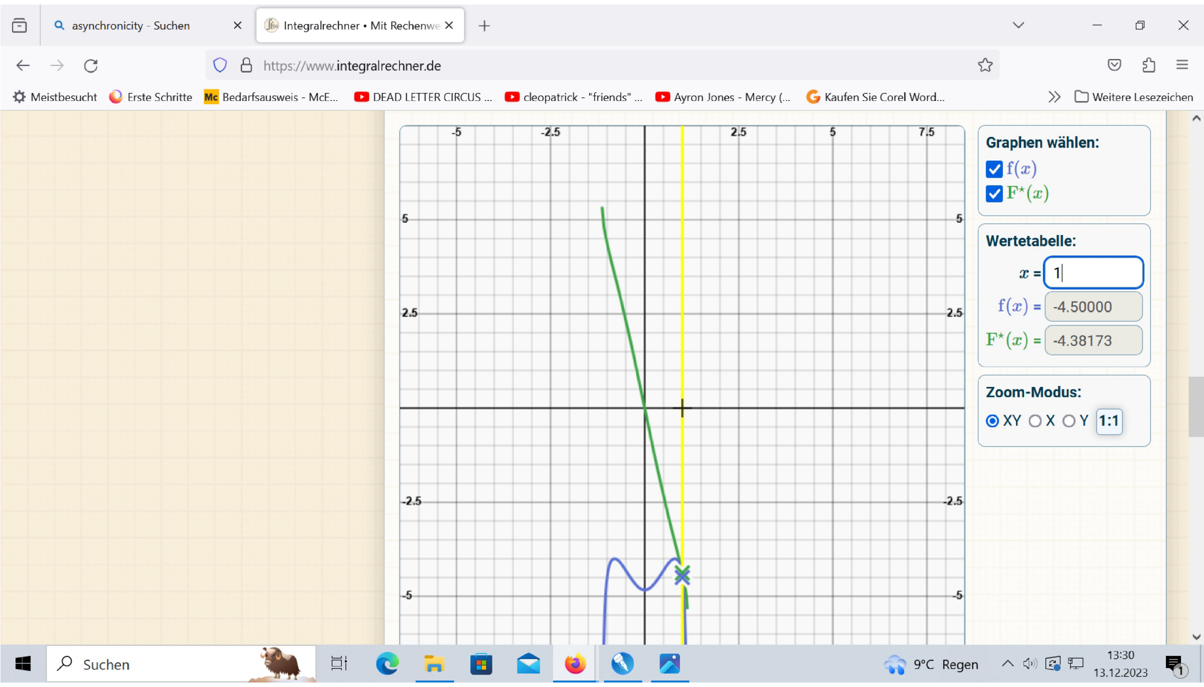

Fig.3: A **t,r**-diagram showing the world line of Alice (traveling almost at the speed of light) who is traversing a spherical body from **r**(=**x**)=+1 to **r**(=**x**)= -1 with $\mathbf{r_s/R_0}$=7/9 (generated online at www.integralrechner.de).The local

escape velocity is below **c** everywhere (with $\mathbf{R_0}$=1).

More precisely: Alice is smoothly traveling into Bob's future when she is traversing the spherical body, as it may take 1000000 or so years in Bob's time **t** for Alice to cover the spatial distance between $\mathbf{r}=+\mathbf{R_0}=+1$ and $\mathbf{r}=-\mathbf{R_0}=-1$. In Alice's proper time **tau**, this leg of her trip takes only some moments of time.

b) As a consequence, having traversed the spherical body (which no longer constitutes a Black Hole), Alice could no longer find that time in Bob's frame of reference is dilated, that is, has been elapsing much slower with respect to her own (see below for an explanation why Alice was justified in expecting that to happen). But given time in another frame of reference is compressed relative to one's own time, one cannot assert that one is at rest, that is, one cannot assert to be at the center of an inertial system. For a frame of reference to be an inertial system, Special and General Relativity require that any other frame of reference, inertial or non-inertial, is either dilated in time with respect to one's own, or has the same rate of temporal flow.

The above shall be explained in greater detail: When Alice (who is traveling at a speed of 0.999998 **c** in Bob's frame of reference, but hasn't yet come close to the spherical body she will be traversing) contemplates some clocks that are at rest in Bob's unaccelerated frame of reference and are positioned along Alice's future path, she concludes that those of Bob's clocks whose momentary positions are still far away from her run ahead of time compared to those of Bob's clocks which Alice is passing by momentarily (Alice's rushing past this clock shall be called point-event 1). This is true despite the fact that all of Bob's clocks are synchronized in his (Bob's) frame of reference. Alice expects that, when eventually passing by one of those distant clocks (Alice's rushing past that clock shall be called point-event 2), she will be realizing that the hands of the formerly distant clock will have moved forward only a tiny little bit, whereas the hands of her own clock will have moved forward to a much larger extent between these two point-events. In other words: Alice expects that Bob's time is dilated. That is what Special Relativity tells her.

But in case Alice's traverse of the spherical body would take a million years or so in Bob's time, Alice would find that the hands of Bob's formerly distant clock have moved forward (between the two point-events) to a much larger extent than the hands of her own clock have. In other words: She would have to state a *compression* of Bob's time, and not a dilation.

Any *compression* of time (seen in the other reference frame) is not reciprocal. That is to say: In Bob's frame of reference, Alice's time is *dilated*; and it would even be more dilated if it took a million or so years in Bob's time for Alice to traverse the spherical body. A reciprocity can only be achieved if time is *dilated* (and not *compressed*) in the other system. This has been made obvious already: In Alice's frame of reference, those clocks (all of which are stationary in Bob's frame of reference) that Alice was to pass at a later point of her time ran ahead of those clocks she was passing by momentarily. This asynchronicity did not and does not exist in Bob's frame of reference. The ambiguity of synchronicity or asynchronicity of clocks that are spatially separated from each other is the deeper cause of the compatibility of Alice's as well as Bob's right to say that time is dilated in the other system. It's impossible, however, to do the same with a *compression* rather than a *dilation* of time in the other system.

With no reciprocity between reference frames regarding time, one reference frame is privileged over the other in some way. But all reference frames that constitute inertial systems, that is, reference frames in which an observer is allowed to consider himself or herself at rest, do have this property of reciprocity, that is, they lack of any privileges regarding time in comparison with other frames. As a consequence, the reference frame from the perspective of which time in another frame of reference is compressed cannot be an inertial system.

This is why Relativity would not allow Alice to consider herself as having been at rest, that is, at the center of an inertial system, all the time. This applies even to cases in which $\mathbf{r_s}/\mathbf{R_0}<8/9$, that is, to cases in which the spherical body (which Alice is traversing) does not constitute a Black Hole, not even near the center.

c) On the other hand, General Relativity would get into serious trouble if Alice were *not* allowed to consider herself at rest during her whole trip. This shall be demonstrated immediately.

General Relativity expanded the reach of the physical relativity principle as follows: Any observer who does not feel an accelerating force is allowed to consider himself or herself at rest (and things around him or her in accelerated or unaccelerated motion), even if he or she is, when looked at from a different frame of reference, in free radial fall or rise within a gravity field. A. Einstein (2018) described this broadened contents of the relativity principle as follows (in a hardly known statement):

"*At that moment I got the happiest thought of my life in the following form: In an example worth considering, the gravitational field has a relative existence only in a manner similar to the electric field generated by magneto-electric induction. Because for an observer in free-fall from the roof of a house there is during the fall—at least in his immediate vicinity—no gravitational field. Namely, if the observer lets go of any bodies, they remain relative to him, in a state of rest or uniform motion, independent of their special chemical or physical nature. The observer, therefore, is justified in interpreting his state as being 'at rest.' The extremely strange and confirmed experience that all bodies in the same gravitational field fall with the same acceleration immediately attains, through this idea, a deep physical meaning. Because if there were just one single thing to fall in a gravitational field in a manner different from all others, the observer could recognize from it that he is in a gravitational field and that he is falling. But if such a thing does not exist—as experience has shown with high precision—then there is no objective reason for the observer to consider himself as falling in a gravitational field. To the contrary, he has every right to consider himself in a state of rest and his vicinity as free of fields as far as gravitation is concerned. The experimental fact that the acceleration in free-fall is independent of the material, therefore, is a powerful argument in favor of expanding the postulate of relativity to coordinate systems moving nonuniformly relative to each other.*"

This postulate is implicitly contained in Einstein's field equation of General Relativity.

4) How the dilemma is resolved

Consequently, Alice's clearly "visible" leap into the past in case 1>=$\mathbf{r_s/R_0}$>=8/9 (see Fig. 1) was necessary in order to make up for the dilation of time (viewed from Bob's perspective) which Alice had experienced in the vicinity of the spherical mass (as an effect of the gravitational field). If uncompensated, that dilation would have led to a "forbidden" compression of Bob's time if viewed from Alice's perspective.

[A similar dilemma and its solution present themselves in the realm of the *outer* Schwarzschild solution, in which the mass of the gravitating spherical body is concentrated at **r**=0, or, when considering the cosmic variant (De-Sitter-space), in which there is no mass at all. See Trupp (2020) and Trupp (2024).]

In case $\mathbf{r_s/R_0}$<8/9 , the situation is not qualitatively different. In order to save the relativity principle, a tunneling of Bob's time at **r=0** by means of Alice's leap into Bob's past must still occur, even though the spherical body does no longer constitute a Black Hole.

Equation (3) does not stand in the way of observing this. Having presupposed that infalling Alice reaches **r**=0 at Bob's time **t**=0, the first line of (3) simply has to be modified as follows: (7)

$$t_{r_0} = t_{r_1} + \Delta t - T = 0$$

The summand **T** is the amount of Bob's time tunneled by Alice at **r**=0 for the sake of the relativity principle of General Relativity. **T** is different from zero even if Alice traversed a spherical body like planet Earth (with $\mathbf{r_s/R_0}$ being very small). **T** is zero whenever the spherical body is massive enough so that the infalling or outgoing object has to overcome a barrier at which the local escape velocity is **c**.

5) Summary of results

The relativity principle in General Relativity, according to which any observer in free radial fall or rise within a gravity field is allowed to consider himself or herself at rest, requires that an interval of Bob's coordinate time is tunneled by a photon or fast falling Alice whenever one of them is traversing a spherical body for which $\mathbf{r_s/R_0}$ is <= 8/9.

With regard to spherical bodies for which 1>= $\mathbf{r_s/R_0}$ >8/9, the inner Schwarzschild solution yields a world line of falling Alice with a singularity, and a time-reversed section of the world line between the singularity and the center of the sphere. In these cases, the integral of **dt** (that determines the shape of the world line from $\mathbf{R_0}$ all the way to the center) is not divergent between +$\mathbf{R_0}$ and **r**=0, nor is it divergent between **r**=0 and -$\mathbf{R_0}$.

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