# The “photon sphere” around Black Holes as a mere artefact

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**Abstract: It is shown that the photon sphere (region of $r_s < r < 1.5\ r_s$) where stable orbits shall be impossible (since the orbital velocity would have to exceed c although the radial escape velocity does not) is a mere artefact. It is the result either of a flawed application of the Euler-Lagrange equation (insofar as the wrong frame of reference is used for determining the kinetic energy of orbital motion), or of a flawed application of the equation for a r-geodesic (insofar as it is wrongly presupposed that an orbit is a geodesic). Thereby the postulate by P. Graneau and N. Graneau regarding the centrifugal force as a true and not just fictitious force is proven to be correct.**

## 1) Introduction

The “photon sphere” outside of the Schwarzschild radius of Black Holes where stable orbits are supposed to be impossible has always been a counter-intuitive postulate. This paper is to prove that it does not exist.

## 2) The orbit determined by the Euler-Lagrange equation in Newtonian physics

**a)** When applying Newtonian mechanics for deriving the equation for an orbit, we take the Euler-Lagrange equation
(1)

$$\frac{\delta L}{\delta r} = \frac{d}{dt}\frac{\delta L}{\delta v}$$

as a starting-point.

We consider the moment in time at which a satellite on its circular orbit in the **x,y**-plane of a three-dimensional Cartesian coordinate system (**x,y,z**) crosses the positive **x**-axis. The center of coordinates coincides with the center of the central spherical mass. The orbital velocity shall be determined entirely according to Newton’s mechanics.

For the Langrangian **L** (with **T** being the kinetic energy, with **V** being the potential energy of the orbiter, **m** being its mass and **M** being the mass of the gravitating body) we get:
(2)

$$L = T - V = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) + \frac{mGM}{r}$$

Hence, when using *Cartesian coordinates*, the Euler-Lagrange equation turns into (for the moment at which the orbiter crosses the positive **x**-axis)

(3)

$$-\frac{mGM}{r^2} = -\frac{mGM}{x_0^2} = \frac{d}{dt}\,mv_x + \frac{d}{dt}\,mv_y + \frac{d}{dt}\,mv_x = \frac{d}{dt}\,mv_x$$

For geometrical reasons, and if time **t** is chosen to be exactly zero when the orbiter crosses the positive **x**-axis (at $\mathbf{x=x_0}$), we get for the differential change in the **x**-coordinate of the orbiter:
(4)

$$-dx = x_0 - x_0 \cos \omega t$$

The second derivative of the **x**-coordinate with respect to time **t** thus gives:
(5)

$$\frac{d}{dt}\,v_x\ (for\ t=0) = \frac{d^2x}{dt^2} = -x_0\,\omega^2 \cos \omega t = -x_0\,\omega^2 = -\frac{v_{orbit}^2}{x_0}$$

From (3) and (4) we finally get:
(6)

$$v_{orbit}^2 = \frac{GM}{x_0} = \frac{GM}{r} = \frac{c^2 r_s}{2r}$$

**b)** If polar coordinates are used (instead of Cartesian coordinates), the Lagrangian **L** reads as follows:
(7)

$$L = T - V = \frac{m}{2}\left(\frac{dr^2}{dt^2} + r^2\frac{d\varphi^2}{dt^2}\right) + \frac{GMm}{r}$$

The Euler-Lagrange equation thus converts into two equations (see *L. Susskind/G. Hrabowsky*,[1] pp. 218-220, Eq. 9 and 6), the first one being:
(8)

$$-\frac{d^2r}{dt^2} = \frac{GM}{r^2} - r\,\frac{d\varphi^2}{dt^2}$$

(8) postulates: The second derivative of the polar coordinate **r** with respect to time (that has no uniquely determined direction), multiplied by -1, is equal to the numerically positive gravitational force plus the numerically negative centrifugal force.

The second equation is:
(9)

$$\frac{d}{dt}\left(mr^2\frac{d\varphi}{dt}\right) = 0$$

It says that that the angular momentum is conserved.

On a circular orbit, the second derivative of the polar coordinate **r** (that has no uniquely determined direction) with respect to time is zero, but only because the force of gravity and the centrifugal force are equal and opposite to each other. We therefore get from (8):
(10)

$$\frac{d\varphi^2}{dt^2} = \frac{GM}{r^3}$$

### 3) Adapting the result to the Schwarzschild metric

**a)** When it comes to adapting this Newtonian equation (10) or (6) to the Schwarzschild metric, all we have to do is to add a prime to **dt**, so that the equation shows the situation as it presents itself for a local observer at rest in the gravity field at distance **r** from the center of the spherical body. Hence we get for the local observer at rest in the gravity field:
(11)

$$\frac{d\varphi^2}{(dt')^2} = \frac{GM}{r^3}$$

In other words: One has to realize that the Euler-Lagrange equation is nothing but an expression of d'Alembert's principle. But this principle must be understood as a local one. That is to say: one has to put oneself in the position of a local observer at rest in the gravity field when applying it. In the reference frame of that observer, the gravitational force is counteracted by an opposing force, this force being the centrifugal force. That constitutes the contents of the Euler-Lagrange equation if applied to an orbit. From the perspective of an observer who sits outside of the gravity field, laws of nature – for instance that of the invariance of the speed of light – are not always valid at places inside the gravity field.

**b)** In order to switch to the reference frame of a distant (stationary) observer outside the gravity field, **dt'²** (the time of the local observer at rest in the gravity field) has to be replaced by $\mathbf{dt^2(1\text{-}r_s/r)}$ in accordance with the Schwarzschild metric. We then get from (11):
(12)

$$\frac{d\varphi^2}{dt^2} = (1 - \frac{r_s}{r})\frac{GM}{r^3} = (1 - \frac{r_s}{r})\frac{c^2 r_s}{2r^3}$$

According to (12), orbits at $\mathbf{r<1.5\,r_s}$ are perfectly possible.

### 4) An apparent discrepancy between different ways of applying the Euler-Lagrange equation

**a)** Nevertheless, there is an apparent discrepancy between the result provided by polar coordinates on the one hand, and by Cartesian coordinates on the other hand, no matter if we consider Newton's original mechanics or its modification by the Schwarzschild metric: When using polar coordinates, the second derivative of **r** with respect to time is zero, whereas, when using Cartesian coordinates, the second derivative of **x** with respect to time is different from zero, though **x** is equal to **r** at the moment in time considered. In other words: There is an anti-radial, that is, centripetal force of magnitude $md^2x/dt^2$ (in the direction of the center) when using Cartesian coordinates, whereas there is no such force when using polar coordinates.

The discrepancy vanishes as soon as one realizes that, if using polar coordinates, the second derivative of (undirectional) **r** with respect to time (which is zero) yields the anti-radial (positive) gravitational force plus the (negative) centrifugal force (see above).

**b)**There is another apparent discrepancy between two ways of applying the Euler-Lagrange equation. Based on the Schwarzschild solution for **d tau**, one may postulate that the principle of least action holds true in the following form (**L** is the Lagrangian; see also R.P. Feynman[(2)], Eq. 7.3.2 and 7.3.3, p. 96/97):
(13)

$$d\int_1^2 L dt = d\int_1^2 d\tau = d\int_1^2 \sqrt{(1-\frac{r_s}{r} - \frac{dr^2}{c^2(1-r_s/r)dt^2} - r^2\frac{d\varphi^2}{c^2dt^2})}\ dt = 0$$

This requires that the square root is an expression of the kinetic energy minus the potential energy. But even if $1-r_s/r$ could be proportional to the potential energy (per kg of mass) for a local observer at rest in the gravity field and thus be an expression of it, the third term cannot be the kinetic energy of orbital motion in the reference frame of the local observer, for his/her time is **t'** and not **t**. The situation is different from purely radial motion, when the third term is zero and the second term is not (because of the relativistic factor, the second term *is* an expression of the kinetic energy in the reference frame of the stationary local observer). Hence, the Euler-Lagrange equation and the principle of least action cannot be applied if that special **L** is used for orbital motion, since those equations require the usage of the kinetic and potential energies of a test body (orbiter) as they present themselves in the frame of a *local* observer.

That error can be found in several textbooks [see for instance: N. Dragon [(3)], Equation 6.19, derived from 6.17, which leads to 6.21 and 6.24 on pp. 124, 125; the error can be traced back to J. Droste [(4)], p. 203, Equation 17, which is the result of the derivation of his Lagrangian with respect to **r** ].

## 5) The common (but erroneous) derivation of the equation for a circular orbit by means of the r-geodesic

**a)** In order to determine the orbital velocity of a body revolving around a non-rotating Black Hole on a perfect circle in the equatorial plane of a reference frame with polar coordinates,

some authors resort to the equation of the **r**-geodesic (see for this well known method: V. Frolov, A. Zelnikov [(5)], p. 185). This equation reads:
(14)

$$\frac{d^2R}{d\tau^2} + \Gamma^1_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \frac{d^2R}{d\tau^2} + \Gamma^r_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

The variable **x** (of which there are four, that is three for spatial coordinates and one for the temporal coordinate) denotes the coordinates of an event in spacetime, and runs from $\mathbf{x^0}$ to $\mathbf{x^3}$. Since polar coordinates are used, we have: $\mathbf{x^0= t, x^1= r, x^2=}$ **theta,** $\mathbf{x^3 =}$ **phi .**

Time **t** is the time of a distant observer at rest, whereas **tau** is the time of an observer coasting along the geodesic in the gravity field. The angle **theta** is the angle from North Pole to the equator; **phi** is the azimutal angle. The distance **r** is circumference divided by 2 **pi**.

The formula for computing the Christoffel symbols [that appear in Equation (14)] is:
(15)

$$\Gamma^p_{\mu\nu} = \frac{g^{pn}}{2}\left(\frac{\delta g_{n\mu}}{\delta x^\nu} + \frac{\delta g_{n\nu}}{\delta x^\mu} - \frac{\delta g_{\mu\nu}}{\delta x^n}\right)$$

Since the motion of the body is confined to the equatorial plane, **theta** is 90° and is fixed. The metric tensor **g** (in polar coordinates) is derived from the Schwarzschild metric. This is because the right-hand side of the general equation
(16)

$$d\tau^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

(where the products **gdxdx** have to be summed over **mu** und **nu**, which are both running from 0 to 3) has to be equal to the right side of equation of the Schwarzschild metric for a Black Hole.

If the second derivative of **R** or **r** with respect to **tau** is supposed to be zero (Delta **R** is radial distance between two points along a radial line measured by the number of stationary meter sticks laid end to end, whereas **r** is circumference divided by 2 pi), the formula of the **r**-geodesic gives:
(17)

$$\Gamma^1_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} = \left(1-\frac{r_s}{r}\right)\frac{c^2 r_s}{2r^2}\frac{dt^2}{d\tau^2}$$

$$= -\Gamma^1_{33}\frac{dx^3}{d\tau}\frac{dx^3}{d\tau} - \Gamma^1_{11}\frac{dx^1}{d\tau}\frac{dx^1}{d\tau} = \left(1-\frac{r_s}{r}\right) r\,\frac{d\varphi^2}{d\tau^2} + \left[\frac{r_s}{2(1-r_s/r)\,r^2}\right]\frac{dr^2}{d\tau^2}$$

All other products of Christoffel-symbols (denoted by capital Gamma) and their appropriate

**dx/dtau** times **dx/dtau** vanish.

Multiplication of (17) by **dtau²/dt²** , **1/r** , **1/(1-r$_s$/r)**, and a re-arrangement lead to:
(18)

$$\frac{d\varphi^2}{dt^2} = \frac{c^2 r_s}{2r^3} - \frac{r_s}{2(1-r_s/r)^2 r^3}\,\frac{dr^2}{dt^2}$$

If **dr/dt** is set to zero, one gets from (18):
(19)

$$\frac{d\varphi^2}{dt^2} = \frac{c^2 r_s}{2r^3}$$

This formula (19) for a perfectly circular orbit can be found in many textbooks (see for instance Ch.W. Misner, K.S. Thorne, J.A. Wheeler [(6)], § 25.5, Equation 25.40, p. 668). It differs from the correct equation (12) by a relativistic factor.

**b)** But although the second derivative of a non-directional **r** or **R** (as a polar coordinate) with respect to proper time **tau** of an orbiter is zero in the Euler-Langrange equation (see above), it may not be permissible to set it to zero in equation (14) of the **r**-geodesic: (14) simply states that for a test body to be travelling along a geodesic, any acceleration it is subject to is the result of a change in the curvature of spacetime, and is not the (co-)result of a force exerted on it. But an orbiter is subject to centrifugal force, and it was because the sum of the (negative) centrifugal force plus the (positive) centripetal force was zero that we were allowed to set **d²R/ d tau²= 0** in the Euler-Lagrange equation. Hence, before setting **d²R/d tau²** to zero in (14), we must find out if the centrifugal force is a real force or a mere fictitious force in GR, or, to put it differently, if an orbit constitutes a geodesic. Only if it does, can the **r**-geodesic be utilized to derive the equation of orbital motion.

This investigation is done in the following: In case **d²R/ dtau²** is not set to zero, but is left as it is, Equation (14) of a geodesic does no longer give (17), but gives:
(20)

$$\frac{d^2R}{d\tau^2} + \left(1 - \frac{r_s}{r}\right)\frac{c^2 r_s}{2r^2}\,\frac{dt^2}{d\tau^2} = \left(1 - \frac{r_s}{r}\right) r\,\frac{d\varphi^2}{d\tau^2} + \frac{r_s}{2\left(1 - \frac{r_s}{r}\right)r^2}\,\frac{dr^2}{d\tau^2}$$

When multiplying both sides of (20) by **d tau ²/dt²** , **1/r** and **(1-r$_s$/r)$^{-1}$**, (20) converts into:
(21)

$$\frac{d^2R}{d\tau^2}\,\frac{d\tau^2}{dt^2}\,\frac{1}{r(1 - r_s/r)} = -\frac{c^2 r_s}{2r^3} + \frac{r_s}{2(1-r_s/r)^2 r^3}\,\frac{dr^2}{dt^2} + \frac{d\varphi^2}{dt^2}$$

If we set **dr/dt=0**, and leave **dphi/dt** as it is, (21) converts into:
(22)

$$\frac{d^2R}{d\tau^2}\frac{d\tau^2}{dt^2}\frac{1}{r(1-r_s/r)} = -\frac{c^2r_s}{2r^3} + \frac{d\varphi^2}{dt^2}$$

From this follows by re-arrangement of (22) (the proper time **tau** of a stationary observer in the gravity field at distance **r** shall be replaced by **t' ,** with **tau** now being the proper time of the orbiter; **$d^2R/dt'^2$** is replaced by **$-c^2r_s/2r^2$** , that is by the gravity according to Newton, see below):
(23)

$$\frac{d\varphi^2}{dt^2} = \frac{c^2r_s}{2r^3} + \frac{d^2R}{d\tau^2}\frac{d\tau^2}{dt^2}\frac{1}{r(1-r_s/r)} = \frac{c^2r_s}{2r^3} + \frac{d^2R}{r\ (dt')^2} = \frac{c^2r_s}{2r^3} - \frac{c^2r_s}{2r^3} = 0$$

This demonstrates that a circular orbit is not a geodesic. In other words: For the circular orbit to be a geodesic, the angular velocity must be zero. But then there is no orbit at all. This is why (23) describes a moment at which an object that has been thrown radially upward comes to a standstill at its climax, in order to fall back down thereafter. As a consequence (see above), the centrifugal force is not a mere fictitious force in GR! This is what P. Graneau and N. Graneau [(5)] (p. 113 and p. 156/157) had advanced in 2006.

A confirmation is achieved when scrutinizing the **phi**-geodesic:
(23a)

$$\frac{d^2x^3}{d\tau^2} + \Gamma^3_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \frac{d^2\varphi}{d\tau^2} + \Gamma^\varphi_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

It turns out that every single summand is zero. That is to say: No equation for the determination of **dphi/dtau** can be derived from (23a):
(23b)

$$\Gamma^3_{00}\frac{dt^2}{d\tau^2} = [\frac{g^{30}}{2}(...) + \frac{g^{31}}{2}(...) + \frac{g^{32}}{2}(...) + \frac{g^{33}}{2}(\frac{\delta g_{30}}{...} + \frac{\delta g_{30}}{...} - \frac{\delta g_{00}}{\delta x^3})]\frac{dt^2}{d\tau^2} = 0$$

$$\Gamma^3_{11}\frac{dr^2}{d\tau^2} = [....]\frac{dr^2}{d\tau^2} = 0$$

$$\Gamma^3_{22}\frac{d\theta^2}{d\tau^2} = [....]\frac{d\theta^2}{d\tau^2} = 0$$

$$\Gamma^3_{33}\frac{d\varphi^2}{d\tau^2} = [\frac{g^{30}}{2}(...) + \frac{g^{31}}{2}(...) + \frac{g^{32}}{2}(...) + \frac{g^{33}}{2}(\frac{\delta g_{33}}{\delta x^3} + \frac{\delta g_{33}}{\delta x^3} - \frac{\delta g_{33}}{\delta x^3})]\frac{d\varphi^2}{d\tau^2} = 0$$

$$\Gamma^3_{03}\frac{dt}{d\tau}\frac{d\varphi}{d\tau} = [\frac{g^{30}}{2}(...) + \frac{g^{31}}{2}(...) + \frac{g^{32}}{2}(...) + \frac{g^{33}}{2}(\frac{\delta g_{30}}{...} + \frac{\delta g_{33}}{\delta x^0} - \frac{\delta g_{03}}{...})]\frac{dt}{d\tau}\frac{d\varphi}{d\tau} = 0$$

$$\Gamma^3_{30}\frac{d\varphi}{d\tau}\frac{dt}{d\tau} = [\frac{g^{30}}{2}(...) + \frac{g^{31}}{2}(...) + \frac{g^{32}}{2}(...) + \frac{g^{33}}{2}(\frac{\delta g_{33}}{\delta x^0} + \frac{\delta g_{30}}{...} - \frac{\delta g_{30}}{...})]\frac{d\varphi}{d\tau}\frac{dt}{d\tau} = 0$$

$$\Gamma^3_{01}\frac{dt}{d\tau}\frac{dr}{d\tau} = [....]\frac{dt}{d\tau}\frac{dr}{d\tau} = 0$$

$$\Gamma^3_{10}\frac{dr}{d\tau}\frac{dt}{d\tau} = [....]\frac{dr}{d\tau}\frac{dt}{d\tau} = 0$$

$$\Gamma^3_{20}\frac{d\theta}{d\tau}\frac{dt}{d\tau} = [....]\frac{d\theta}{d\tau}\frac{dt}{d\tau} = 0$$

$$\Gamma^3_{02}\frac{dt}{d\tau}\frac{d\theta}{d\tau} = [....]\frac{dt}{d\tau}\frac{d\theta}{d\tau} = 0$$

$$\Gamma^3_{21}\frac{d\theta}{d\tau}\frac{dr}{d\tau} = [....]\frac{d\theta}{d\tau}\frac{dr}{d\tau} = 0$$

$$\Gamma^3_{12}\frac{dr}{d\tau}\frac{d\theta}{d\tau} = [....]\frac{dr}{d\tau}\frac{d\theta}{d\tau} = 0$$

$$\Gamma^3_{13}\frac{dr}{d\tau}\frac{d\varphi}{d\tau} = [....]\frac{dr}{d\tau}\frac{d\varphi}{d\tau} = 0$$

$$\Gamma^3_{31}\frac{d\varphi}{d\tau}\frac{dr}{d\tau} = [....]\frac{d\varphi}{d\tau}\frac{dr}{d\tau} = 0$$

$$\Gamma^3_{23}\frac{d\theta}{d\tau}\frac{d\varphi}{d\tau} = [....]\frac{d\theta}{d\tau}\frac{d\varphi}{d\tau} = 0$$

$$\Gamma^3_{32}\frac{d\varphi}{d\tau}\frac{d\theta}{d\tau} = [....]\frac{d\varphi}{d\tau}\frac{d\theta}{d\tau} = 0$$

**c)** This solves a problem that arises in the context of the “twin paradox”: When a spaceship that was accelerated to its final speed of almost **c** passes planet Earth, the clocks on board and on Earth are synchronized. When reaching a distant star, the spaceship follows a slingshot trajectory around that star as Apollo 13 did around the moon. Having returned to the solar system (and passing by the Earth once more), the lag of the spaceship's clocks (that constitute the "twin paradox") is absolute, and not relative, just as if the spaceship had fired its engine at its destination in order to return to the solar system. But if the slingshot trajectory had been force-free, its effect could not be the same as a firing of the rocket engine, and its crew would have the same right to expect that Earthlings' clocks lag behind as Earthlings have with respect to the crew's clocks.

**d)** Hence, one runs into various contradictions when applying (19) to orbits, especially those that are close to the Schwarzschild radius. For instance, when replacing the angular velocity by ordinary velocity **v**, (19) turns into:
(24)

$$v^2 = \frac{c^2 r_s}{2r}$$

For $\mathbf{r=r_s}$, we find:
(25)

$$v_{orbital} = \sqrt{\frac{c^2}{2}} = 0.7\ c$$

But this contradicts earlier findings according to which any motion at the Schwarzschild radius is "frozen" from the perspective of a far-away observer.

This is why all textbook that support (24) cannot but claim that no stable orbit can exist below a certain radius that is determined by the following reflection: If, in the equation of the Schwarzschild metric, proper time **tau** of a light pulse is set to zero, the equation of the Schwarzschild metric (for a light pulse in a circular orbit around a spherical, massive body) turns into:
(26)

$$\frac{d\varphi^2}{dt^2} = \frac{c^2}{r^2}\,(1-\frac{r_s}{r})$$

Combining (26) with (19) gives:
(27)

$$\frac{d\varphi^2}{dt^2} = \frac{c^2}{r^2}\,(1-\frac{r_s}{r}) = \frac{c^2 r_s}{2r^3}$$

or
(28)

$$r = \frac{3}{2} r_s$$

This is the radius of the so-called "photon sphere" commonly found in textbooks. From the perspective of a stationary observer sitting at r= 1.5$\mathbf{r_s}$, the local velocity **v'** of any orbiting object is **c**.

But with the local orbital velocity $\mathbf{v'_{orbital}}$ at 1.5$\mathbf{r_s}$ thus postulated to be higher than the local radial escape velocity $\mathbf{v'_{escape}}$, it is implicitly postulated that the inert mass of a testbody in the gravity field is lower than its heavy mass if the testbody is in orbital motion (so that it needs a higher orbital velocity in order to generate a centrifugal force that matches the gravitational force in magnitude). This would constitute a clear violation of the equivalence principle, and is therefore incompatible with GR.

**e)** Equation (23) was based on the presumtion that, in GR, gravitational acceleration experienced by a local observer in the gravity field is equal to Newton's. The validity of this assertion shall be proven:

When setting **dphi/dt=0**, (21) turns into:
(29)

$$\frac{d^2R}{d\tau^2}\,\frac{d\tau^2}{dt^2}\,\frac{1}{r(1-r_s/r)} = -\frac{c^2 r_s}{2r^3} + \frac{r_s}{2(1-r_s/r)^2 r^3}\,\frac{dr^2}{dt^2}$$

Since, according to the Schwarzschild metric, **d tau²/dt²** times $\mathbf{(1\text{-}r_s/r)^{-1}}$ equals unity if **tau** is the time of a (at least momentarily) stationary observer in the gravity field, and since **dr/dt** is zero with respect to a (at least momentarily) stationary object that had been thrown upward and has now reached its climax, we get for this situation from (29):
(30)

$$\frac{d^2R}{d\tau^2} = -\frac{c^2 r_s}{2r^2} = -\frac{MG}{r^2}$$

This is Newton's law of gravity – valid for a local observer at rest in the gravity field.
(30) can be turned into:
(31)

$$\frac{d^2R}{dt^2} = -\frac{c^2 r_s}{2r^2}\left(1-\frac{r_s}{r}\right) = -\frac{MG}{r^2}\left(1-\frac{r_s}{r}\right)$$

In order to make sure that the choice of **d²R** (rather than **d²r**) in (14) and hence in (29), (30) and (31) is justified, let us re-consider (29). That equation can be re-arranged in order to present itself as follows, given that **tau** is the time of an observer at rest in the gravity field, **dr/dt** is the velocity of an object in free fall in the frame of an observer outside the gravity field, **v'** is the velocity of that object in the frame of the second observer who is at rest in the gravity field, and who is watching the object passing by:
(32)

$$\frac{d^2R}{d\tau^2} = \left[-\frac{c^2 r_s}{2r^2} + \frac{r_s}{2(1-r_s/r)^2 r^2}\frac{dr^2}{dt^2}\right]\left(1 - \frac{r_s}{r}\right)\frac{dt^2}{d\tau^2} = \left[-\frac{c^2 r_s}{2r^2} + \frac{(v')^2 r_s}{2r^2}\right]$$

**$(1\text{-}r_s/r)^{-2}$ $dr^2/dt^2$** is equal to **$v'^2$** (radial velocities are affected both by the contraction of meter sticks and by the dilation of time), and **$(1\text{-}r_s/r)$ $dt^2/d$ tau²** is equal to unity.

According to (32), an object whose radial velocity is **c** does not experience any more radial acceleration. Consequently, radial velocities that are below **c** in the beginning cannot exceed the local speed of light outside of the Schwarzschild radius. Moreover, (32) shows that even far away from the gravitating mass, that is at a large – though not infinite – distance **r** from the mass, an object cannot be in possession of a velocity greater than **c** to start with: If the velocity **v'** of an object at a large distance r were greater than **c**, the acceleration **$d^2R/d$ tau²** would be positive, meaning that the object would be decelerating (though only slightly if **v'** is not much greater than **c**) because of a reversal of the gravitational "force". But this could not be a physically valid statement. Instead, one has to conclude that velocities greater than **c** are physically impossible even in almost flat spacetime, that is, in Special Relativity.

The fact that, according to (32), **$d^2R/d$ tau²** is always negative or zero has a further consequence: Imagine we would have written **$d^2r/$ d tau²** instead of **$d^2R/d$ tau²** in (14) and hence in (32). Then (32) would require that **$d^2r/d$ tau²,** which can also be written as **$(1\text{-}r_s/r)^{-1}d^2r/dt^2$,** has to be always negative (given **$v'^2<c^2$**). But since a freely falling object, when watched from outside of the gravity field, is gathering speed only to start with, and is slowing down near the Schwarzschild radius, **$d^2r/dt^2$** cannot always be negative [whereas **$(1\text{-}r_s/r)^{-1}$** is always positive]. Instead, **$d^2r/dt^2$** has to change sign from negative to positive somewhere between the starting point of the free fall and the Schwarzschild radius. In order to avoid this inconsistency, **$d^2R$,** and not **$d^2r$,** had to be used in (14) and in (29) to (32).

Notes and References

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