

Tegmark's mathematical universe vindicated

by Andreas Trupp

Abstract: Leaving aside the values of constants, Einstein's field equations of General Relativity and hence the Schwarzschild-Droste metric (that comprises both the metric of Black Holes and also of flat Minkowski spacetime) are derived mathematically from a simple empirical assumption only, namely that the mass of a non-rotating body at rest does not change over time. Moreover, even this empirical assumption has a mathematical part, since it is supported by a symmetry argument. Tegmark's hypothesis of a mathematical universe is thereby corroborated.

I.

The derivation of Einstein's master field equations of General Relativity (in its contravariant form)

(1)

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi G T^{\mu\nu}$$

can be accomplished in the following way:

When considering the left hand side and the middle part (and assuming the whole equation is the result of pure speculation), it follows by mere mathematical reasoning that their covariant divergence is zero (the tensor \mathbf{G} is the Einstein-tensor, \mathbf{R} with superscripts denotes the Ricci-tensor, the scalar \mathbf{R} is the contracted Ricci-tensor, \mathbf{g} is the metric tensor, the scalar \mathbf{G} is Newton's constant). That is to say: The correctness of

(2)

$$\nabla_{\mu}G^{\mu\nu} = \nabla_{\mu} \left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R \right) = 0$$

follows apriori (as is well known), without any reference to experiments.

For we can formulate:

(2a)

$$\nabla_{\mu}R^{\mu\nu} = \frac{1}{2}g^{\mu\nu} \delta_{\mu}R$$

and can thus replace the covariant divergence of the \mathbf{R} -tensor by an ordinary divergence of the contracted Ricci-tensor.

Moreover, since the covariant divergence of any metric tensor \mathbf{g} is zero, we can formulate:

(2b)

$$\nabla_{\mu} \frac{1}{2} g^{\mu\nu} R = \frac{1}{2} (\nabla_{\mu} g^{\mu\nu}) R + \frac{1}{2} g^{\mu\nu} \nabla_{\mu} R = \frac{1}{2} g^{\mu\nu} \nabla_{\mu} R = \frac{1}{2} g^{\mu\nu} \delta_{\mu} R$$

Subtracting the very left side of (2b) from the left side of (2a) gives zero. Equation (2) is thereby proved to be correct.

II.

1) As regards the right hand side of Einstein's equations, its covariant divergence is not zero a priori, but can only be zero according to empirical observation.

But what is the empirical basis of an assumed zero (covariant) divergence of the energy-momentum tensor \mathbf{T} ? If we restrain ourselves to considering a spherical, non-rotating mass at rest, the energy-momentum tensor (which we could call mass-momentum tensor, instead) \mathbf{T} on the right hand side of Einstein's equation reduces to

(3)

$$T^{\mu\nu} = \begin{matrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

inside the spherical mass (with \mathbf{rho} denoting mass density), and reduces to zero outside.

The ordinary divergence of the energy-momentum tensor is zero, if the density \mathbf{rho} of the spherical mass does not change over time ($\mathbf{d rho/ dt=0}$, empirical assumption). This is because we then have for the interior of the spherical mass:

$$\delta_{\mu} T^{\mu\nu} = 0$$

or

(4)

$$\delta_{\mu} T^{\mu 0} = \frac{\delta T^{00}}{\delta x_0} + \frac{\delta T^{10}}{\delta x_1} + \frac{\delta T^{20}}{\delta x_2} + \frac{\delta T^{30}}{\delta x_3} = \frac{\delta \rho}{\delta t} + \frac{\delta 0}{\delta x_1} + \frac{\delta 0}{\delta x_2} + \frac{\delta 0}{\delta x_3} = 0$$

$$\delta_{\mu} T^{\mu 1} = \frac{\delta T^{01}}{\delta x_0} + \frac{\delta T^{11}}{\delta x_1} + \frac{\delta T^{21}}{\delta x_2} + \frac{\delta T^{31}}{\delta x_3} = \frac{\delta 0}{\delta t} + \frac{\delta 0}{\delta x_1} + \frac{\delta 0}{\delta x_2} + \frac{\delta 0}{\delta x_3} = 0$$

$$\delta_{\mu} T^{\mu 2} = \frac{\delta T^{02}}{\delta x_0} + \frac{\delta T^{12}}{\delta x_1} + \frac{\delta T^{22}}{\delta x_2} + \frac{\delta T^{32}}{\delta x_3} = \frac{\delta 0}{\delta t} + \frac{\delta 0}{\delta x_1} + \frac{\delta 0}{\delta x_2} + \frac{\delta 0}{\delta x_3} = 0$$

$$\delta_{\mu} T^{\mu 3} = \frac{\delta T^{03}}{\delta x_0} + \frac{\delta T^{13}}{\delta x_1} + \frac{\delta T^{23}}{\delta x_2} + \frac{\delta T^{33}}{\delta x_3} = \frac{\delta 0}{\delta t} + \frac{\delta 0}{\delta x_1} + \frac{\delta 0}{\delta x_2} + \frac{\delta 0}{\delta x_3} = 0$$

For locations outside the spherical mass, $\mathbf{d\rho/dt}$ in the first line of (4) is replaced by $\mathbf{d0/dt}$, so that the (ordinary) divergence is again zero.

But it is the *covariant* divergence (and not just the ordinary divergence) that has to be zero. In other words: We should have

(5)

$$\nabla_{\mu} T^{\mu\nu} = \delta_{\mu} T^{\mu\nu} + \Gamma_{\alpha\mu}^{\mu} T^{\alpha\nu} + \Gamma_{\alpha\mu}^{\nu} T^{\mu\alpha} = \Gamma_{\alpha\mu}^{\mu} T^{\alpha\nu} + \Gamma_{\alpha\mu}^{\nu} T^{\mu\alpha} = 0$$

It can be shown that this divergence, too, is indeed zero.

Proof:

It is well known from (3) that all components of \mathbf{T} except \mathbf{T}^{00} (inside the spherical mass) are zero. From this it can be concluded that only the case of $\mathbf{nu=0}$ has to be given consideration in (5). Moreover, the coefficient alpha has no other value than zero. Hence, the right hand side of (5) is zero if:

(6)

$$\Gamma_{\alpha\mu}^{\mu} T^{00} + \Gamma_{\alpha 0}^{\nu} T^{00} = \Gamma_{0\mu}^{\mu} T^{00} + \Gamma_{00}^0 T^{00} = (\Gamma_{00}^0 + \Gamma_{01}^1 + \Gamma_{02}^2 + \Gamma_{03}^3) T^{00} + \Gamma_{00}^0 T^{00} = 0$$

Dividing by T^{00} gives:

(7)

$$\Gamma_{00}^0 + \Gamma_{01}^1 + \Gamma_{02}^2 + \Gamma_{03}^3 + \Gamma_{00}^0 = 0$$

as a condition for the covariant divergence of \mathbf{T} to be zero.

2) a) Next, the equation

(8)

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$$

shall be scrutinized. It presupposes two observers: One observers (Alice) measures spatial and temporal distances between two events in units of \mathbf{dx} ; that coordinate system of Alice shall have \mathbf{x}^0 for time, and $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3$ for the three spatial coordinates. The second observer (Bob) measures the same distance simply in units of \mathbf{ds} (called proper distance). The distance

ds may either be the short spatial distance between two events that occur simultaneously for Bob, or ds may alternatively be the short *temporal* distance between two events that occur at the same spatial location for Bob.

The factor g , which is the metric tensor, has to be determined yet.

Now, let the distance between Bob and Alice have shrunk to zero, and let the relative speed of the two observers be zero. We then get for the metric tensor valid at the location of Bob and Alice, judged in a Cartesian reference frame (in which Alice is at rest):

(9)

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

b) It will soon be realized why this determination of the metric tensor g is well justified even without any empirical corroboration. When re-writing (8) by using (9), we get:

(9a)

$$ds^2 = dx^0 dx^0 + dx^1 dx^1 + dx^2 dx^2 + dx^3 dx^3$$

aa) Let us, as a first of two alternatives, define that ds is a *spatial* distance for Bob. (9a) then turns into:

(9b)

$$ds^2 = dx^1 dx^1 + dx^2 dx^2 + dx^3 dx^3$$

(9b) is valid under the assumption that dx^0 is zero. But this is not an empirical assumption: Stating that Bob and Alice – who find themselves at rest at the same location in space – both judge the temporal distance between two nearby events as being zero follows from the symmetry of the situation.

As one can easily realize, (9b) is an expression of the Pythagorean theorem, of which we know without any empirical test, namely by mere by geometric reflections, that it is correct in space of any kind, given the nearby volume of space considered is sufficiently small.

bb) As the second alternative, let us define that ds is a *temporal* distance for Bob. (9a) then converts into:

(9c)

$$ds^2 = dx^0 dx^0$$

(9c) states that the distance in time between the two nearby events considered is measured equally long by Alice and Bob. The correctness of this statement is again provided by the

symmetry of the situation.

Moreover, (9c) states that the zero spatial distance between the two events which is presupposed for Bob is also a zero spatial distance for Alice. But this, too, follows from symmetry reflections, and does not need any empirical corroboration.

c) Any partial derivative of this metric tensor \mathbf{g} (expressed by Equation 9) with respect to $\mathbf{x}^0(=\mathbf{t})$, \mathbf{x}^1 , \mathbf{x}^2 or \mathbf{x}^3 is zero. The zero results of any partial derivative of \mathbf{g}_{00} or of any other component of \mathbf{g} are an indirect expression of the mathematical recognition that any surface, space or spacetime is flat, if the area (the volume, respectively) considered is small enough.

3) Consequently, given the definition of a Christoffel symbol, i.e.
(10)

$$\Gamma_{\mu\nu}^{\rho} = \frac{g^{\rho n}}{2} \left(\frac{\delta g_{n\mu}}{\delta x^{\nu}} + \frac{\delta g_{n\nu}}{\delta x^{\mu}} - \frac{\delta g_{\mu\nu}}{\delta x^{\rho}} \right)$$

each of the Christoffel symbols in (7) is zero.

Thereby it is being proved that the covariant divergence of the (special) energy-momentum tensor \mathbf{T} is zero (inside and outside the spherical mass) in a local, Cartesian reference frame. But if the covariant divergence of the tensor \mathbf{T} is zero in the Cartesian frame of reference, the covariant divergence of \mathbf{T} is, as a mathematical necessity, zero in any other frame of reference as well (in which the spherical mass is stationary and does not change over time).

III.

With both the covariant divergence of the energy momentum-tensor \mathbf{T} (the components of which are zero with the exception of \mathbf{T}^{00}) and also the covariant divergence of the Einstein-tensor \mathbf{G} – which is short-hand for the middle part of (1) – being zero, we are confronted with a situation in which the four four-vectors $\mathbf{T}^{0\mu}$, $\mathbf{T}^{1\mu}$, $\mathbf{T}^{2\mu}$, $\mathbf{T}^{3\mu}$ on the one hand (the index μ runs from 0 to 3, so each of the four \mathbf{T} -terms represents four numbers, i.e., the components of a four-vector), and the four four-vectors $\mathbf{G}^{0\mu}$, $\mathbf{G}^{1\mu}$, $\mathbf{G}^{2\mu}$, $\mathbf{G}^{3\mu}$ on the other hand are each divergenceless (see Equation 7 as regards \mathbf{T} -vectors). In general, that does *not* entail that $\mathbf{T}^{0\mu}$ is equal to $\mathbf{G}^{0\mu}$, or that $\mathbf{T}^{1\mu}$ is equal to $\mathbf{G}^{1\mu}$. The \mathbf{T} -field lines may differ in structure from the corresponding \mathbf{G} -field.

But the two tensors share common properties, if we can make sure that all eight vectors (those tensors consist of in total), if not zero, have the same direction. The direction of the only non-zero \mathbf{T} -vector, that is the four vector $\mathbf{T}^{0\mu}$ (whose components are $\mathbf{T}^{00}=\mathbf{rho}$, $\mathbf{T}^{01}=\mathbf{0}$, $\mathbf{T}^{02}=\mathbf{0}$, $\mathbf{T}^{03}=\mathbf{0}$) is strictly in the time-direction. Now, for reasons of symmetry, and since all \mathbf{G} -vectors have a zero divergence, the only directions in which (non-zero) \mathbf{G} -vectors may point in a three-dimensional $\mathbf{t},\mathbf{x},\mathbf{y}$ -diagram are the time direction on the one hand, and circles around the \mathbf{t} -axis, that is, circles in the \mathbf{x},\mathbf{y} -plane on the other hand. But in a four-dimensional $\mathbf{t}, \mathbf{x}, \mathbf{y}, \mathbf{z}$ -diagram, circles may exist not only in the \mathbf{x},\mathbf{y} -plane, but also in the \mathbf{x},\mathbf{z} -plane, the \mathbf{y},\mathbf{z} -plane or in parallel planes. However, since there is no privileged direction, one orientation

of the Cartesian reference frame and hence of the circles would be as good as any other. Thus, for reasons of symmetry, there can be no closed, circular field lines (representing \mathbf{G} -vectors that make up the \mathbf{G} -tensor) at all. Nor can a \mathbf{G} -vector-field form three-dimensional spheres with the mass at the center. This is because purely spherical fields cannot exist as vector fields (but only as scalar fields). There is no defined direction for a vector to point at.

This, in turn, entails that the $\mathbf{T}^{0\mu}$ -vector (with the component \mathbf{T}^{00} being equal to ρ , and the other three components being zero) and the $\mathbf{G}^{0\mu}$ -vector point in the same direction (all the other \mathbf{G} - and \mathbf{T} -vectors that make up the \mathbf{G} - and the \mathbf{T} -tensor are zero).

We thus know that the two tensors \mathbf{G} and \mathbf{T} are distinguished from each other only by a scalar factor, since the only non-zero component of \mathbf{G} , that is $\mathbf{G}^{0\mu}$, forms a vector that points in the same direction as does the vector $\mathbf{T}^{0\mu}$ which forms the only non-zero component of \mathbf{T} . That is to say: Dividing the \mathbf{G} -tensor by the \mathbf{T} -tensor gives a scalar, whose magnitude is equal to the quotient of the magnitude of the non-zero component of \mathbf{G} , that is $\mathbf{G}^{0\mu}$, divided by the magnitude of the non-zero component of \mathbf{T} , that is $\mathbf{T}^{0\mu}$.

Unfortunately, the scalar quotient of the magnitudes of the $\mathbf{G}^{0\mu}$ and $\mathbf{T}^{0\mu}$ vectors, though not depending on time and though pointing in the same direction, cannot (yet) be said to be surely a constant. This is because that quotient may – at first sight – depend on the distance \mathbf{r} from the center of the spherical mass. Moreover, it may approach zero or infinity.

We can nevertheless state:

(10a)

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \frac{|G^{0\mu}|}{|T^{0\mu}|} T^{\mu\nu}$$

Equation (10a), which we have arrived at on the basis of a very simple empirical assumption (namely, that the mass of a body left to its own devices does not change), differs from Einstein's master equations (1) only by the scalar factor in front of the tensor \mathbf{T} .

IV.

1) Astonishingly, the nature of the scalar factor can be determined still further without performing experiments.

We know that heavy bodies in a gravity field fall at the same rate as do light bodies. In his "Dialogue Concerning Two New Sciences" ("Discorsi") of 1638 (translated by H. Crew and A. de Salvio, Macmillan 1914, p. 62/63), Galileo Galilei presented an argument in favour of the hypothesis that a heavy body does not move faster in a gravitational field than a light body does, "without further experiment", that is by pure reasoning alone: If a heavy body and a light body are connected to each other by a cord, and if the heavy body is undergoing a greater acceleration than the light one, the cord will be under strain. Since a cord under strain would exert force on an object to which it is tied, the heavy body would feel a force that is reducing its speed of free fall. Consequently, the combination of the two bodies would

undergo a weaker acceleration than the heavy body alone would. But on the other hand, the combination of the two bodies can be regarded as being one single body that is heavier than the heavy body alone is. Consequently, the combination of the two bodies should undergo a stronger acceleration than the heavy body alone would. But the combination of the two bodies cannot, at the same time, undergo a stronger and a weaker acceleration than the heavy body alone would.

The only way to avoid a contradiction is to assume that the two bodies undergo exactly the same acceleration. (Before Galilei, it was Giovanni Battista Benedetti who had presented this line of thought in his “Demonstratio proportionum motuum localium” of 1554.)

That is to say: From the concept of force, that is, from the definition that force accelerates or decelerates an object, it is deduced by pure reasoning, i.e., without experiment, that all bodies in free fall are subject to exactly the same acceleration.

2) Consequently, any replacement of the scalar factor in (10a) by a better-defined factor must lead to the result that all bodies undergo the same acceleration in the gravity field regardless of their mass.

If, by pure speculation, the scalar factor appearing in front of the tensor \mathbf{T} in (10a) is assumed to be equal to a constant \mathbf{k} whose magnitude is $8 \pi \mathbf{G}$ (with \mathbf{G} being Newton’s constant), we arrive at (1).

From this equation (Einstein’s field equations of General Relativity), the equation of the Schwarzschild(-Droste)-metric, that is
(11)

$$d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \frac{1}{c^2 \left(1 - \frac{2GM}{c^2 r}\right)} dr^2 - \frac{r^2}{c^2} (d\theta^2 + \sin^2\theta d\varphi^2)$$

is obtained by purely mathematical reasoning (as is well known). The term $2GM/c^2$ can be replaced by the Schwarzschild radius r_s .

3) We shall now find out whether or not (according to Equation 11) all bodies undergo the same acceleration in a gravity field, regardless of their mass.

If, in the formula for a \mathbf{r} -geodesic, that is
(12)

$$\frac{d^2 R}{d\tau^2} + \Gamma_{\mu\nu}^1 \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \frac{d^2 r}{d\tau^2} + \Gamma_{\mu\nu}^r \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0$$

$d^2 \mathbf{x}^1 / d\tau^2$ is exchanged for $d^2 \mathbf{r} / d\tau^2$, and if the Christoffel symbol is written in full detail on the basis of the Schwarzschild metric, we get:

(13)

$$\frac{d^2R}{d\tau^2} \frac{d\tau^2}{dt^2} \frac{1}{r(1 - r_s/r)} = - \frac{c^2 r_s}{2r^3} - \frac{r_s}{2(1-r_s/r)^2 r^3} \frac{dr^2}{dt^2} - \frac{d\phi^2}{dt^2}$$

R stands for the radial distance measured by laying meter sticks end to end; **r** stands for the circumference of a circle around the spherical mass, divided by **2 pi**.

Since, according to the Schwarzschild metric, **dtau**²/**dt**² times **(1-r_s/r)**⁻¹ equals unity if **tau** is the time of a (at least momentarily) stationary observer in the gravity field, and since with respect to a stationary object both **dr/dt** and **dphi/dt** are zero, we get for this situation from (13):

$$\frac{d^2R}{d\tau^2} = - \frac{c^2 r_s}{2r^2} = - \frac{MG}{r^2}$$

or:

(13a)

$$\frac{d^2R}{dt^2} = - \frac{c^2 r_s}{2r^2} \left(1 - \frac{r_s}{r}\right)$$

(13a) is identical with the result obtained by Droste in 1917 (page 203).

M is the mass of the spherical body, **G** is Newton's gravitational constant.

This proves that – on the basis of the scalar factor being a constant **k** (and not a variable) – all bodies undergo the same acceleration in the local gravity field regardless of the magnitude of their masses.

It is thus evident that **G^{0_μ / T^{0_μ}}** (as an absolute number) appearing in (10a) can be equal to the constant **k** that appears in (1). Assuming that there is no other equation than that of the Schwarzschild-droste metric [presented in (11)] which meets said requirements (though that cannot be proved here), the validity of Einstein's master equation (1) is hereby demonstrated without the need for any further empirical proof.

V.

1) In other words: It has been made evident that the physical validity of the Schwarzschild-Droste metric – which is derived from (1) – is based on a single empirical recognition only, namely that the mass of a non-rotating spherical body at rest does not change over time. From this (and the concept of force) alone, all the properties of spacetime outside a spherical mass (described by the Schwarzschild-Droste metric) follows as a purely mathematical

consequence.

Moreover, for r much larger than the Schwarzschild radius $2GM/c^2$, the Schwarzschild metric converts into the Minkowski metric of flat spacetime (in polar coordinates). To put it differently: It does not need the invariance of the (local) speed of light and the classical relativity principle to get to the Minkowski metric and hence to the Lorentz-transformation of Special Relativity. These two laws, and therefore Special Relativity, can be derived from the single and simple empirical assumption just mentioned, instead.

Hence, when eclipses of the sun were observed in 1919 and later on, it was not only Einstein's theory of General Relativity that was subject to empirical testing: If the outcome of the observations had been different from that predicted by the Schwarzschild-Droste metric, it would have been proved that the mass of an object like the sun is not constant over time, or that k – as a constant – would have a value different from what had been assumed.

2) Moreover, the basic empirical assumption (of an invariance of the spherical mass) is not purely empirical in character. Its correctness is backed by a symmetry argument, whose character, like that of all symmetry arguments, is mathematical: If the density of the stationary mass varied considerably out of the blue, these moments in time would be special compared to other moments in time, though no reason for such a special role would be detectable.

3) This is strong support for Max Tegmark's hypothesis of a mathematical universe. All the complex properties of space and time in the vicinity of a gravitating spherical mass including the properties of a non-rotating Black Hole are derived – by purely mathematical reasoning and not by further observations or experiments – from the simple empirical recognition just mentioned.

We conclude: Nature is not just *described* by mathematics, nature *is* mathematics, instead.

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Appendix:

Galileo Galilei, Dialogue Concerning Two New Sciences (“Discorsi”), 1638 (translated by H. Crew and A. de Salvio, Macmillan 1914), p. 62/63:

“SALV. But, even without further experiment, it is possible to prove clearly, by means of a short and conclusive argument, that a heavier body does not move more rapidly than a lighter one provided both bodies are of the same material and in short such as those mentioned by Aristotle. But tell me, Simplicio, whether you admit that each falling body acquires a definite (63)

speed fixed by nature, a velocity which cannot be increased or diminished except by the use of force [violenza] or resistance.

SIMP. There can be no doubt but that one and the same body moving in a single medium has a fixed velocity which is determined by nature and which cannot be increased except by the addition of momentum [impeto] or diminished except by some resistance which retards it.

SALV. If then we take two bodies whose natural speeds are different, it is clear that on uniting the two, the more rapid one will be partly retarded by the slower, and the slower will be somewhat hastened by the swifter. Do you not agree with me in this opinion?

SIMP. You are unquestionably right.

SALV. But if this is true, and if a large stone moves with a speed of, say, eight while a smaller moves with a speed of four, then when they are united, the system will move with a speed less than eight; but the two stones when tied together make a stone larger than that which before moved with a speed of eight. Hence the heavier body moves with less speed than the lighter; an effect which is contrary to your supposition. Thus you see

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how, from your assumption that the heavier body moves more rapidly than the lighter one, I infer that the heavier body moves more slowly.”