Tegmark's mathematical universe vindicated

by Andreas Trupp

Abstract: *Leaving aside the values of constants, Einstein's field equations of General Relativity and hence the Schwarzschild-Droste metric (that comprises both the metric of Black Holes and also of flat Minkowski spacetime) are derived mathematically from a simple empirical assumption only, namely that the mass of a non-rotating body at rest does not change over time. Moreover, even this empirical assumption has a mathematical part, since it is supported by a symmetry argument. Tegmark's hypothesis of a mathematical universe is thereby corroborated.*

I.

The derivation of Einstein's master field equations of General Relativity (in its contravariant form)

(1)

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi G\ T^{\mu\nu}$$

can be accomplished in the following way:

When considering the left hand side and the middle part (and assuming they have been placed there by pure guessing), it follows by mere mathematical reasoning that its covariant divergence is zero (the tensor **G** is the Einstein-tensor, **R** with superscripts denotes the Ricci-tensor, the scalar **R** is the contracted Ricci-tensor, **g** is the metric tensor, the scalar **G** is Newton's constant). That is to say: The correctness of

(2)

$$\nabla_\mu G^{\mu\nu} = \nabla_\mu\ (R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R) = 0$$

follows apriori (as is well known), without any reference to experiments. For we have

$$\nabla_\mu R^{\mu\nu} = \frac{1}{2}g^{\mu\nu}\ \delta_\mu R$$

and (since the covariant divergence of any metric tensor **g** is zero)

$$\nabla_\mu\ \frac{1}{2}g^{\mu\nu}R = \frac{1}{2}(\nabla_\mu g^{\mu\nu})R + \frac{1}{2}g^{\mu\nu}\ \nabla_\mu R = \frac{1}{2}g^{\mu\nu}\ \nabla_\mu R = \frac{1}{2}g^{\mu\nu}\ \delta_\mu R$$

Equation (2) is thereby proved to be correct.

II.

1) As regards the right hand side of Einstein's equations (again, we are assuming it has been placed there by pure guessing), its covariant divergence is not zero apriori, but can only be zero according to empirical observation.

But what is the empirical basis of an assumed zero (covariant) divergence of the energy-momentum tensor **T**? If we restrain ourselves to considering a spherical, non-rotating mass at rest, the energy-momentum tensor (which we could call mass-momentum tensor, instead) **T** on the right hand side of Einstein's equation reduces to
(3)

$$T^{\mu\nu} = \begin{matrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

inside the spherical mass (with **rho** denoting mass density), and reduces to zero outside.

The ordinary divergence of the energy-momentum tensor is zero, if the density **rho** of the spherical mass does not change over time (**d rho/ dt=0**, empirical assumption). This is because we then have for the interior of the spherical mass:

$$\delta_\mu T^{\mu\nu} = 0$$

or
(4)

$$\delta_\mu T^{\mu 0} = \frac{\delta T^{00}}{\delta x_0} + \frac{\delta T^{10}}{\delta x_1} + \frac{\delta T^{20}}{\delta x_2} + \frac{\delta T^{30}}{\delta x_3} = \frac{\delta \rho}{\delta t} + \frac{\delta 0}{\delta x_1} + \frac{\delta 0}{\delta x_2} + \frac{\delta 0}{\delta x_3} = 0$$

$$\delta_\mu T^{\mu 1} = \frac{\delta T^{01}}{\delta x_0} + \frac{\delta T^{11}}{\delta x_1} + \frac{\delta T^{21}}{\delta x_2} + \frac{\delta T^{31}}{\delta x_3} = \frac{\delta 0}{\delta t} + \frac{\delta 0}{\delta x_1} + \frac{\delta 0}{\delta x_2} + \frac{\delta 0}{\delta x_3} = 0$$

$$\delta_\mu T^{\mu 2} = \frac{\delta T^{02}}{\delta x_0} + \frac{\delta T^{12}}{\delta x_1} + \frac{\delta T^{22}}{\delta x_2} + \frac{\delta T^{32}}{\delta x_3} = \frac{\delta 0}{\delta t} + \frac{\delta 0}{\delta x_1} + \frac{\delta 0}{\delta x_2} + \frac{\delta 0}{\delta x_3} = 0$$

$$\delta_\mu T^{\mu 3} = \frac{\delta T^{03}}{\delta x_0} + \frac{\delta T^{13}}{\delta x_1} + \frac{\delta T^{23}}{\delta x_2} + \frac{\delta T^{33}}{\delta x_3} = \frac{\delta 0}{\delta t} + \frac{\delta 0}{\delta x_1} + \frac{\delta 0}{\delta x_2} + \frac{\delta 0}{\delta x_3} = 0$$

For locations outside the spherical mass, **d rho/dt** in the first line of (4) is replaced by **d0/dt**, so that the (ordinary) divergence is again zero.

But it is the *covariant* divergence (and not just the ordinary divergence) that has to be zero. In other words: We should have
(5)

$$\nabla_\mu T^{\mu\nu} = \delta_\mu T^{\mu\nu} + \Gamma^\mu_{\alpha\mu} T^{\alpha\nu} + \Gamma^\nu_{\alpha\mu} T^{\mu\alpha} = \Gamma^\mu_{\alpha\mu} T^{\alpha\nu} + \Gamma^\nu_{\alpha\mu} T^{\mu\alpha} = 0$$

It can be shown that this divergence, too, is indeed zero.

Proof:

It is well known from (3) that all components of **T** except $\mathbf{T^{00}}$ (inside the spherical mass) are zero. From this it can be followed that only the case of **nu=0** has to be given consideration in (5). Moreover, the coefficient alpha has no other value than zero. Hence, the right hand side of (5) is zero if:
(6)

$$\Gamma^\mu_{\alpha\mu} T^{00} + \Gamma^\nu_{\alpha 0} T^{00} = \Gamma^\mu_{0\mu} T^{00} + \Gamma^0_{00} T^{00} = (\Gamma^0_{00} + \Gamma^1_{01} + \Gamma^2_{02} + \Gamma^3_{03}) T^{00} + \Gamma^0_{00} T^{00} = 0$$

Dividing by T^{00} gives:
(7)

$$\Gamma^0_{00} + \Gamma^1_{01} + \Gamma^2_{02} + \Gamma^3_{03} + \Gamma^0_{00} = 0$$

as a condition for the covariant divergence of **T** to be zero.

2) Now, let the differential **ds** either be the short spatial distance between two events, measured by a stationary observer who is standing close by these events (and let these two events be simultaneous for that observer), or let **ds** be the short *temporal* distance between two events, measured by him or her who is standing close by these events (and let those two events occur at the same location for him or her). Finally, let us introduce a second stationary observer, who is located some unknown distance away from the first one. For a short while, the distance between the two observers shall be undetermined and may range from zero to infinity. The coordinate system of the second observer shall have $\mathbf{x^0}$ for time, and $\mathbf{x^1}$,$\mathbf{x^2}$, $\mathbf{x^3}$ for the three spatial coordinates. We then have (as a very general geometric statement):
(8)

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

The factor **g**, which is the metric tensor, has to be determined yet.

The first stationary observer mentioned above shall be positioned inside the non-rotating spherical mass. Let the distance between this observer and the second one shrink to zero

because the second observer has moved to the spatial position of the first one. When using a Cartesian coordinate system, we get for the metric tensor valid at the position inside the spherical mass, judged in the Cartesian reference frame of the second observer at that position:
(9)

$$g_{\mu\nu} = \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

It will soon be realized why this determination of the metric tensor **g** is well justified: When re-writing (8) by using (9), and given **ds** is the spatial or temporal distance between two events that are chosen to occur simultaneously in the reference frame of the second observer (so that $\mathbf{dx^0=0}$), we get:

$$ds^2 = dx^1dx^1 + dx^2dx^2 + dx^3dx^3$$

Since $\mathbf{dx^1}$, $\mathbf{dx^2}$ and $\mathbf{dx^3}$ can by choice have units of length and not of time, it follows that **ds** is spatial and not temporal distance in the reference frame of the first observer. Hence we are confronted with the Pythagorean theorem, of which we know by geometric reflections that it is physically correct if the volume of space considered (in which both observers find themselves) is sufficiently small.

If two events are – in the Cartesian coordinate system of the second observer – chosen to occur at the same place, but at different times, and given (9) is valid, (8) converts into:

$$ds^2 = dx^0dx^0$$

Since $\mathbf{dx^0}$ can by choice have units of time, it follows that **ds** constitutes a temporal (and not a spatial) distance in the reference frame of the first observer. We could call that temporal distance **dtau** instead of **ds**. The result of an equality of $\mathbf{dx^0(=dt)}$ and **dtau(=ds)** is justified by the fact that the two Cartesian reference frames of the two observers are indistingushable from each other, given that there is no relative motion of the two observers and hence of the two Cartesian reference frames.

Any partial derivative of this metric tensor **g** with respect to $\mathbf{x^0(=t)}$, $\mathbf{x^1}$, $\mathbf{x^2}$ or $\mathbf{x^3}$ is zero: Though $\mathbf{g_{00}}$ (which is an expression of how the rate of ticking of a clock in the hands of the second observer differs from the rate of ticking of a clock in the hands of the first observer), valid for the fixed location inside the spherical mass, may depend on the spatial position of the second observer, $\mathbf{dg_{00}/dr}$, judged in the reference frame of the second stationary observer (but with respect to the location of the first observer), is nevertheless zero for the second observer if he or she is positioned right at the spatial location of the first observer (this fact is given its mathematical expression by $\mathbf{g_{00}}$ being a fixed number, and not a term that would contain any of the three spatial variables $\mathbf{x^1}$,$\mathbf{x^2}$ or $\mathbf{x^3}$).

The zero results of any partial derivative of $\mathbf{g_{00}}$ or of any other component of **g** are an indirect expression of the mathematical recognition that any surface, space or spacetime is flat, if the area (the volume, respectively) considered is small enough.

This does not imply, though, that $\mathbf{dg_{00}/dr}$ (that refers to the spatial position of the stationary first observer inside the spherical mass, but is judged in the reference frame of a second stationary observer) is vanishingly small also if the second observer is *far away*. For the far-away second observer, $\mathbf{g_{00}}$ may even depend on time.

3) Consequently, given the definition of a Christoffel symbol, i.e.
(10)

$$\Gamma^{p}_{\mu\nu} = \frac{g^{pn}}{2}\left(\frac{\delta g_{n\mu}}{\delta x^{\nu}} + \frac{\delta g_{n\nu}}{\delta x^{\mu}} - \frac{\delta g_{\mu\nu}}{\delta x^{n}}\right)$$

each of the Christoffel symbols in (7) is zero. The symbols are also zero for any location *outside* of the spherical mass (no matter how far away), if the first and second observer share the same location.

Thereby it is being proved that the covariant divergence of the (special) energy-momentum tensor **T** is zero (inside and outside the spherical mass) in a local, Cartesian reference frame. But if the covariant divergence of the tensor **T** is zero in the Cartesian frame of reference, the covariant divergence of **T** is, as a mathematical necessity, zero in any other frame of reference as well (in which the spherical mass is stationary and does not change over time).

III.

With both the covariant divergence of the energy momenum-tensor **T** (the components of which are chosen to be zero with the exception of $\mathbf{T^{00}}$) and also the covariant divergence of the Einstein-tensor **G** – which is short-hand for the middle part of (1) – being zero, we are confronted with a situation in which the four four-vectors $\mathbf{T^{0\mu}}$, $\mathbf{T^{1\mu}}$, $\mathbf{T^{2\mu}}$, $\mathbf{T^{3\mu}}$ on the one hand (the index $\boldsymbol{\mu}$ runs from 0 to 3, so each of the four **T**-terms represents four numbers, i.e., the components of a four-vector), and the four four-vectors $\mathbf{G^{0\mu}}$, $\mathbf{G^{1\mu}}$, $\mathbf{G^{2\mu}}$, $\mathbf{G^{3\mu}}$ on the other hand are each divergenceless (see Equation 7 as regards **T**-vectors). In general, that does *not* entail that $\mathbf{T^{0\mu}}$ is equal to $\mathbf{G^{0\mu}}$, that $\mathbf{T^{1\mu}}$ is equal to $\mathbf{G^{1\mu}}$, and so on, even if a multiplicative constant is introduced. The **T**-field lines may differ in structure from the corresponding **G**-field.

But things look somewhat different if we can make sure that all eight vectors, if not zero, have the same direction. The direction of the only non-zero **T**-vector, that is the four vector $\mathbf{T^{0\mu}}$ (whose components are $\mathbf{T^{00}=rho}$, $\mathbf{T^{01}=0}$, $\mathbf{T^{02}=0}$, $\mathbf{T^{03}=0}$) is strictly in the time-direction. Now, for reasons of symmetry, and since all **G**-vectors have a zero divergence, the only directions in which (non-zero) **G**-vectors may point in a three-dimensional **t,x,y**-diagram are the time direction on the one hand, and circles around the **t**-axis, that is, circles in the **x,y**-plane on the other hand. But in a four-dimensional **t, x, y, z**-diagram, circles may exist not only in the **x,y**-plane, but also in the **x,z**-plane, the **y,z**-plane or in parallel planes. However, since there is no privileged direction, one orientation of the Cartesian reference frame and hence of the circles would be as good as any other. Thus, for reasons of symmetry, there can be no closed, circular field lines (representing **G**-vectors that make up the **G**-tensor) at all.

This, in turn, entails that the $\mathbf{T^{0\mu}}$-vector (with the component $\mathbf{T^{00}}$ being equal to **rho**, and the other three components being zero) and the $\mathbf{G^{0\mu}}$-vector point in the same direction (all the other **G**- and **T**-vectors that make up the **G**- and the **T**-tensor are zero). The scalar quotient of the magnitudes of $\mathbf{G^{0\mu}}$ and $\mathbf{T^{0\mu}}$ vectors, though not depending on time (thereby assuring the existence of a partial proportionality between $\mathbf{G^{0\mu}}$ and $\mathbf{T^{0\mu}}$ at least), may – at first sight – depend on the distance **r** from the center of the spherical mass. Moreover, it may approach zero or infinity.

We may hence state:
(10a)

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \frac{|G^{0\mu}|}{|T^{0\mu}|}\, T^{\mu\nu}$$

IV.

1) Astonishingly, the nature of the scalar quotient can be determined still further without performing experiments. We know that heavy bodies in a gravity field fall at the same rate as do light bodies. In his “Dialogue Concerning Two New Sciences” (“Discorsi”) of 1638 (translated by H. Crew and A. de Salvio, Macmillan 1914, p. 62/63), Galileo Galilei presented an argument in favour of the hypothesis that a heavy body does not move faster in a gravitational field than a light body does, “without further experiment”, that is by pure reasoning alone: If a heavy body and a light body are connected to each other by a cord, and if the heavy body is undergoing a greater acceleration than the light one, the cord will be under strain. Since a cord under strain would exert force on an object to which it is tied, the heavy body would feel a force that is reducing its speed of free fall. Consequently, the combination of the two bodies would undergo a weaker acceleration than the heavy body alone would. But on the other hand, the combination of the two bodies can be regarded as being one single body that is heavier than the heavy body alone is. Consequently, the combination of the two bodies should undergo a stronger acceleration than the heavy body alone would. But the combination of the two bodies cannot, at the same time, undergo a stronger and a weaker acceleration than the heavy body alone would.

The only way to avoid a contradiction is to assume that the two bodies undergo exactly the same acceleration. (Before Galilei, it was Giovanni Battista Benedetti who had presented this line of thought in his “Demonstratio proportionum motuum localium” of 1554.)

From the concept of force, that is, from the recognition that force accelerates or decelerates an object, it is deduced by pure reasoning, i.e., without experiment, that all bodies in free fall are subject to exactly the same acceleration.

2) Consequently, any choice of the scalar quotient must lead to the result that all bodies undergo the same acceleration in the gravity field regardless of their mass.

If, again by pure guessing, the scalar quotient is assumed to be a constant **k** whose magnitude is **8 pi G** (with **G** being Newton's constant), we obtain (1).

From this equation (Einstein's field equations of General Relativity), the equation of the Schwarzschild(-Droste)-metric, that is
(11)

$$d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \frac{1}{c^2\left(1 - \frac{2GM}{c^2 r}\right)} dr^2 - \frac{r^2}{c^2}(d\theta^2 + \sin^2\theta \; d\varphi^2)$$

is obtained by purely mathematical reasoning (as is well known), under the condition that (3) is valid. The term **2GM/c²** can be replaced by the Schwarzschild radius $\mathbf{r_s}$.

3) If, in the formula for a **r**-geodesic, that is
(12)

$$\frac{d^2x^1}{d\tau^2} + \Gamma^1_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \frac{d^2r}{d\tau^2} + \Gamma^r_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

$\mathbf{d^2x^1/dtau^2}$ is exchanged for $\mathbf{d^2r/dtau^2}$, and if the Christoffel symbol is written in full detail on the basis of the Schwarzschild metric, we get:
(13)

$$\frac{d^2r}{d\tau^2}\frac{d\tau^2}{dt^2}\frac{1}{r(1 - r_s/r)} = \frac{c^2 r_s}{2r^3} - \frac{r_s}{2(1-r_s/r)^2 r^3}\frac{dr^2}{dt^2} - \frac{d\varphi^2}{dt^2}$$

Since, according to the Schwarzschild metric, $\mathbf{dtau^2/dt^2}$ times $\mathbf{(1-r_s/r)^{-1}}$ equals unity if **tau** is the time of a (at least momentarily) stationary observer in the gravity field, and since with respect to a stationary object both **dr/dt** and **dphi/dt** are zero, we get for this situation from (13):
(13a)

$$\frac{d^2r}{dt^2} = \frac{c^2 r_s}{2r^2}\left(1 - \frac{r_s}{r}\right)$$

or
(14)

$$\frac{d^2r}{d\tau^2} = \frac{d^2r}{dt^2(1 - r_s/r)} = \frac{c^2 r_s}{2r^2}$$

(13a) is identical with the result obtained by Droste in 1917 (page 203).

This proves that – on the basis of the scalar quotient being a constant **k** – all bodies undergo the same acceleration in the local gravity field regardless of the magnitude of their masses.

It is thus evident that $\mathbf{G^{0\mu}/T^{0\mu}}$ (as an absolute number) appearing in (10a) can be equal to the constant **k** that appears in (1). Assuming that there is no other equation than that of the Schwarzschild-droste metric [presented in (11)] which meets said requirements (though that cannot be proved here), the validity of Einstein's master equations (1) is hereby demonstrated without the need for any further empirical proof.

V.

1) In other words: It has been made evident that the physical validity of the Schwarzschild-Droste metric – which is derived from (1) or from (10a) – is based on a single empirical recognition only, namely that the mass of a non-rotating spherical body at rest does not change over time. From this alone, all the properties of spacetime outside a spherical mass (described by the Schwarzschild-Droste metric) follows as a purely mathematical consequence.

Moreover, for **r** much larger than the Schwarzschild radius $\mathbf{2GM/c^2}$, the Schwarzschild metric converts into the Minkowki metric of flat spacetime (in polar coordinates). To put it differently: It does not need the invariance of the (local) speed of light and the classical relativity principle to get to the Minkowki metric and hence to the Lorentz-transformation of Special Relativity. These two laws, and therefore Special Relativity, can be derived from the single and simple empirical assumption just mentioned, instead.

Hence, when eclipses of the sun were observed in 1919 and later on, it was not only Einstein's theory of General Relativity that was subject to empirical testing: If the outcome of the observations had been different from that predicted by the Schwarzschild-Droste metric, it would have been proved that the mass of an object like the sun is not constant over time, or that **k** as a constant would have a value different from what had been assumed.

2) Moreover, the basic empirical assumption (of an invariance of the spherical mass) is not purely empirical in character. Its correctness is backed by a symmetry argument, whose character, like that of all symmetry arguments, is mathematical: If the density of the stationary mass varied considerably out of the blue, these moments in time would be special compared to other moments in time, though no reason for such a special role would be detectable.

3) This is strong support for Max Tegmark's hypothesis of a mathematical universe. All the complex properties of space and time in the vicinity of a gravitating spherical mass including the properties of a non-rotating Black Hole are derived – by purely mathematical reasoning and not by further obervations or experiments – from the simple, quasi-mathematical recognition just mentioned.

We conclude: Nature is not just *described* by mathematics, nature *is* mathematics, instead.

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Appendix:

Galileo Galilei, Dialogue Concerning Two New Sciences ("Discorsi"), 1638 (translated by H. Crew and A. de Salvio, Macmillan 1914), p. 62/63:

"SALV. But, even without further experiment, it is possible to prove clearly, by means of a short and conclusive argument, that a heavier body does not move more rapidly than a lighter one provided both bodies are of the same material and in short such as those mentioned by Aristotle. But tell me, Simplicio, whether you admit that each falling body acquires a definite
(63)
speed fixed by nature, a velocity which cannot be increased or diminished except by the use of force [violenza] or resistance.

SIMP. There can be no doubt but that one and the same body moving in a single medium has a fixed velocity which is determined by nature and which cannot be increased except by the addition of momentum [impeto] or diminished except by some resistance which retards it.

SALV. If then we take two bodies whose natural speeds are different, it is clear that on uniting the two, the more rapid one will be partly retarded by the slower, and the slower will be somewhat hastened by the swifter. Do you not agree with me in this opinion?

SIMP. You are unquestionably right.

SALV. But if this is true, and if a large stone moves with a speed of, say, eight while a smaller moves with a speed of four, then when they are united, the system will move with a speed less than eight; but the two stones when tied together make a stone larger than that which before moved with a speed of eight. Hence the heavier body moves with less speed than the lighter; an effect which is contrary to your supposition. Thus you see
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how, from your assumption that the heavier body moves more rapidly than the lighter one, I infer that the heavier body moves more slowly."